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LETTER TO THE EDITOR

On quantization of simple harmonic oscillators

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Abstract. A general framework is presented of quantizing the system of a simple harmonic oscillator. Various methods of quantization so far proposed are accommodated here in a unified form. Some remarks are also made on quantization for the case of more than one oscillator.

After Wigner's 1950 paper [1] a number of authors discussed the possibility of generalizing the canonical quantization method by referring, in particular, to the case of a *simple harmonic oscillator* [2–8]. Very recently the problem has been revived from the viewpoint of quantum group, and various forms of q-deformation proposed [9–11]. In this letter we shall show that as far as the system of a single oscillator is concerned, the problem can be formulated in a simple but very general manner, so that all quantization methods so far proposed can be *accommodated there in a unified form*.

The usual, canonical quantization fulfils a number of physical conditions, and when some of them are weakened or lifted, various possibilities of generalization arise. Those conditions are to maintain: (i) the classical form of the equation of motion for the coordinate Q, i.e. $\ddot{Q}+Q=0$ (in units such as $m=\omega=\hbar=1$); (ii) the classical form of the Hamiltonian $H=(P^2+Q^2)/2$ with P being the momentum $P\equiv\dot{Q}$; (iii) the energy spectrum of the form $E_n=E_0+n$ with $n=0,1,2,\ldots$; and (iv) the correspondence-theoretic limit for matrix elements of Q and P. Needless to say, condition (i) is to define the system, and condition (iii) to guarantee the energy-quanta interpretation. Now, for the sake of convenience let us first fix our terminology as follows. A method of quantization (or the resulting energy spectrum) is said to be Bose-like (Fermi-like) if there is no (an) upper bound $n_{\max}$ for n. The Bose-like case is called Wigner quantization if $E_0=$ *an arbitrary positive number* [1], and para-Bose quantization if $E_0=p/2$ with $p=1,2,\ldots$ [12, 13]. Thus the canonical or Bose quantization corresponds to the special case with $E_0=\frac{1}{2}$. For the Fermi-like case, on the other hand, we must have $E_0=-p/2$ with $p=1,2,\ldots$; here the general case is called para-Fermi quantization, and the special case with $E_0=-\frac{1}{2}$ is called Fermi quantization [12, 13]. The above conditions (i)–(iv) are all fulfilled for Wigner quantization, whereas (ii) and (iv) are not fulfilled for para-Fermi quantization. In what follows we shall mainly be concerned with the possibility in which (ii) and (iv) are lifted.

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Let us now formulate our problem in terms of an algebraic system of three operators a, $a^\dagger$ and $N = N^\dagger$, where $a \equiv (Q + iP)/\sqrt{2}$ and $a^\dagger \equiv (Q - iP)/\sqrt{2}$ as usual. For this algebraic system we then require:

(I) $$[a, N] = a \qquad [a^\dagger, N] = -a^\dagger \tag{1}$$

(II) N is given as

$$N = F([a^\dagger, a]_\alpha) \tag{2}$$

or

$$[a^\dagger, a]_\alpha = F^{-1}(N) \equiv G(N) \tag{2'}$$

where G is a real function, including the case of G = constant, and $[A, B]_\alpha \equiv AB + \alpha BA$ with real $\alpha \neq 0$; and

(III) the existence of at least one representation in which the spectrum of N is bounded below. Our algebraic system is thus specified by α and G.

From (I) and (III) it immediately follows that N is of the spectrum $N_n = N_0 + n$, where N_n is the nth eigenvalue and $n = 0, 1, 2, \ldots$. In order to have the energy spectrum of the form as required by (iii) we have only to put $H = \hbar\omega N = N$ where $E_0 = N_0$. The Heisenberg equations for Q and P then lead to the required equation of motion for Q, thereby fulfilling (i). In this case condition (ii) is not in general fulfilled, contrary to the case considered by some other authors [9, 10]. Obviously a, $a^\dagger$ and N play the roles of annihilation, creation and number operators, respectively.

Let us denote the nth eigenstate of N by $|N_n\rangle \equiv |n\rangle$ and assume that the ground state $|0\rangle$ is normalized and non-degenerate. From the 'commutation relation' (2′) it follows that

$$a(a^\dagger)^n|0\rangle = \left[-\sum_{m=0}^{n} (-\alpha)^{m-n} G(m)\right](a^\dagger)^{n-1}|0\rangle \tag{3}$$

where $G(m) \equiv G(N_m)$ and in particular $G(0) = G(N_0)$. By use of (3) it is then easy to show that any eigenstates with eigenvalue N_n can eventually be rewritten as constant $\times (a^\dagger)^n|0\rangle$. This implies that our state vector space is spanned by $|n\rangle$'s with each $|n\rangle$ being non-degenerate.

By taking the expectation values of both sides of (2′) and by suitably choosing the phases of $|n\rangle$ we obtain the matrix elements of a and $a^\dagger$ as follows:

$$\langle n|a|n+1\rangle = \langle n+1|a^\dagger|n\rangle = \sqrt{I(n)} \tag{4}$$

and all other matrix elements = 0, where

$$I(n) \equiv \|a^\dagger|n\rangle\|^2 = \sum_{m=0}^{n} (-1)^m \alpha^{-(m+1)} G(n-m) \geq 0. \tag{5}$$

Thus, the representation of our algebra is determined by fixing the parameter N_0 that is contained in $G(n) = G(N_n)$.

If $I(n) > 0$ for all n, the quantization in Bose-like. Further, if $I(n) > 0$ up to $n = n' - 1$ and $I(n') \leq 0$, we adjust the parameter N_0 so that $I(n') = 0$ is realized; then $I(n) = 0$ for all $n > n'$. In this case the spectrum of N terminates at $n = n_{max} = n'$, hence the quantization is Fermi-like. If neither of the above is the case, such sets of α and G should be discarded.

We now show that all known cases of quantization can be accommodated in our framework.

(a) *Wigner quantization* [1] (Bose-like, $N_0 \geqslant 0$):

$$\alpha = 1 \qquad G(N) = 2N$$
$$I(n) = \begin{cases} n + 2N_0 & \text{for } n = \text{even} \\ n+1 & \text{for } n = \text{odd}. \end{cases} \tag{6}$$

As mentioned above, this includes Bose and para-Bose quantization as special cases [12, 13].

(b) *Para-Fermi quantization* [12, 13] (Fermi-like, $N_0 = -p/2 = -N_{\max}$ with $p = 1, 2, \ldots$):

$$\alpha = -1 \qquad G(N) = 2N$$
$$I(n) = -(n+2N_0)(n+1). \tag{7}$$

Here it is obvious that $I(n)$ with $N_0 \neq -p/2$ is not permissible. Fermi quantization is the special case with $p = 1$.

(c) *q-deformed quantization* (Biedenharn [9]) (Bose-like, $q > 0$, $N_0 = \text{real}$):

$$\alpha = -q^{-1} \qquad G(N) = -q^{-(N+1)}$$
$$I(n) = [n+1]_q q^{-N_0} \tag{8}$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$.

(d) *q-deformed quantization* (Macfarlane [10]) (Bose-like, $q > 0$, $N_0 = \text{real}$):

$$\alpha = -1 \qquad G(N) = [N]_q - [N+1]_q$$
$$I(n) = [n+1+N_0]_q - [N_0]_q. \tag{9}$$

As is evident from (8) and (9), cases (c) and (d) become equivalent only when $N_0 = 0$; this has often been overlooked in the literature.

Further, it is not a difficult matter to invent new kinds of quantization. For example,

(e) *q-deformed Wigner quantization* (Bose-like $q > 0$, $N_0 > 0$):

$$\alpha = 1 \qquad G(N) = [2N]_q$$
$$I(n) = \begin{cases} [n+2N_0]_q\{n+1\}_q & \text{for } n = \text{even} \\ \{n+2N_0\}_q[n+1]_q & \text{for } n = \text{odd} \end{cases} \tag{10}$$

where $\{x\}_q \equiv (q^x + q^{-x})/(q + q^{-1}) = [\frac{1}{2}]_{q^2}[2]_{q^x}$. The special case with $N_0 = p/2$ ($p = 1, 2, \ldots$) may be regarded as q-deformed para-Bose quantization. We remark that our deformation is more general than that of Floreanini and Vinet ([11] and references therein) in that N_0 is not restricted to $p/2$ with $p = 1, 2, \ldots$.

(f) *q-deformed para-Fermi quantization* (Fermi-like, $N_0 = -p/2 = -N_{\max}$ with $p = 1, 2, \ldots$):

$$\alpha = -1 \qquad G(N) = [2N]_q$$
$$I(n) = [-(n+2N_0)]_q[n+1]_q \tag{11}$$

which agrees with the result of [11]. In this case it is also easy to see that N_0 cannot take other values than those specified above.

In all the above cases, excepting (a), the aforementioned conditions (ii) and (iv) are no longer fulfilled. Any of the q-deformed cases reduces, as $q \to 1$, to the corresponding non-deformed case. It is also straightforward to show that the quantization by Greenberg [14], using $aa^\dagger - qa^\dagger a = 1$ with $-1 < q < 1$, and the one by O'Raifeartaigh *et al* [5], using $N = \lambda a^\dagger a + \mu aa^\dagger$ with $\lambda + \mu = 1$, can similarly be discussed within our framework. Lastly let us point out a further possibility of deformation which may be called:

(g) '*T-D cut-off' deformation* (Bose-like, $q > 1$, N_0 = real):

$$\begin{aligned} &\alpha = -q \qquad G(N) = -q^{-N} \\ &I(n) = (n+1)q^{-(N_0+n+1)}. \end{aligned} \tag{12}$$

What is of particular interest here is that $I(n) \to 0$ as $n \to \infty$, hence the contribution from large n is damped. That is to say, such a deformation plays the role of the so-called Tamm–Dancoff cut-off [15]. Incidentally, the present case resembles, in spirit, the idea of Saavedra who considered the possibility of modifying the canonical commutation relation at high energies [16].

As may be easily checked, our operators a and $a^\dagger$ for the general cases can be expressed in terms of the usual Bose operators b and $b^\dagger$ as follows. Taking b and $b^\dagger$ to be operators such that $\langle n|b|n+1\rangle = \langle n+1|b^\dagger|n\rangle = \sqrt{n+1}$, all other matrix elements = 0 and hence $N_b|n\rangle = n|n\rangle$ with $N_b \equiv b^\dagger b = N - N_0$, we can write

$$a = Ub \qquad a^\dagger = b^\dagger U^\dagger \tag{13}$$

where the N_0-dependent operator U is defined by

$$U \equiv \sum_{n=0}^{n_{\max}} \frac{1}{n!}\sqrt{\frac{I(n)}{n+1}}\,(b^\dagger)^n \frac{\sin(\pi(N_b)}{\pi N_b}\,(b)^n. \tag{13$'$}$$

In this connection we note parenthetically that $|n\rangle$, the simultaneous eigenstate of N and N_b, can be expressed in two ways:

$$|n\rangle = [I(0)I(1)\ldots I(n-1)]^{-1/2}(a^\dagger)^n|0\rangle = (n!)^{-1/2}(b^\dagger)^n|0\rangle. \tag{14}$$

The possibility of rewriting (13) for the case of Wigner quantization was first noticed by Boulware and Deser [4].

Most of the properties possessed by the usual oscillator can be generalized accordingly. For example, we can construct coherent states for the operators a and $a^\dagger$.

So far we have been concerned with free oscillators. In the rest of this letter let us make a few remarks on interacting oscillators. First we consider the case of a single, self-interacting oscillator: the Hamiltonian of such a system is given as $H = H_0 + H_{\rm int}$, where $H_0 = \hbar\omega N = N$ as assumed before, and $H_{\rm int} = H_{\rm int}(a, a^\dagger)$. For a and $a^\dagger$, of course, the Heisenberg equations of motion should hold. Now, in evaluating [a or $a^\dagger$, H] we need the (anti)commutator $[a, a^\dagger]_\mp$ unless $H_{\rm int} = H_{\rm int}(N)$. The required commutator is found, from (4), to be

$$[a, a^\dagger]_\mp = I(N) \mp I(N-1) \equiv J^{(\mp)}(N) \tag{15}$$

with

$$I(N) \equiv \sum_{n=0}^{n_{\max}} |n\rangle I(n) \langle n|. \tag{15'}$$

Thus, the Heisenberg equations take, in general, different forms than those of the (classical) Hamilton equations $\dot{a} = (\partial/\partial a)H$, etc. That is to say, in quantum theory some extra form factors $J^{\mp}(N)$ arise as a result of quantization. (Equivalently, the form factors U and $U^{\dagger}$ may be introduced into $H_{\rm int}$ when the change of variables (13) is made therein.)

In order to deal with the case of more than one oscillator interacting with each other we have to know the 'commutation relations' among tthe operators a_j, $a_j^{\dagger}$ ($j = 1, 2, \ldots, f$), where a_j and $a_j^{\dagger}$ are for the jth oscillator. Obviously, the simplest, but certainly mathematically consistent, way to proceed is to assume that for each individual oscillator the commutation relations remain the same as in the case of $f = 1$, i.e. (1) and (2), whereas between different oscillators the corresponding operators simply commute or anticommute:

$$[\hat{a}_j, \hat{a}_k]_{\mp} = 0 \tag{16}$$

where $j \neq k$ and $\hat{a}_j \equiv a_j$ or $a_j^{\dagger}$. However, such a set of commutation relations has the following shortcoming. Except for cases (a) and (b) mentioned above, the formalism does not, in general, remain invariant under a change of variables $a_j \to a_j' = \Sigma_k c_{jk} a_k$: this is against the original spirit of the transformation theory of quantum mechanics. Thus, when applied, for example, to the field theory with local Lagrangian densities, the Heisenberg equations for field operators take, in general, non-local and highly nonlinear forms, and moreover the statistics of the field, which is a consequence of field quantization, will no longer be a universal, but merely a state-dependent property. We need therefore something more sophisticated than (16).

In this respect it is instructive to recall the situation in para-quantization, i.e. cases (a) and (b). For $f = 1$ the operator N is defined by $[a^{\dagger}, a]_{\mp} = 2N$, where the upper (lower) sign corresponds to the para-Fermi (para-Bose) case. Now, it has been known for some time [13] that the last relation and relations (1) are precisely those of the Lie algebra so(3) (Lie superalgebra osp(1/2)). And, to obtain the required commutation relations for the para-Fermi (para-Bose) case with $f > 1$ we have only to enlarge the above algebra [13] to so($2f+1$) (osp($1/2f$)). It is thus hoped that a similar systematic approach will be attempted for the case of q- or more general deformations.

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